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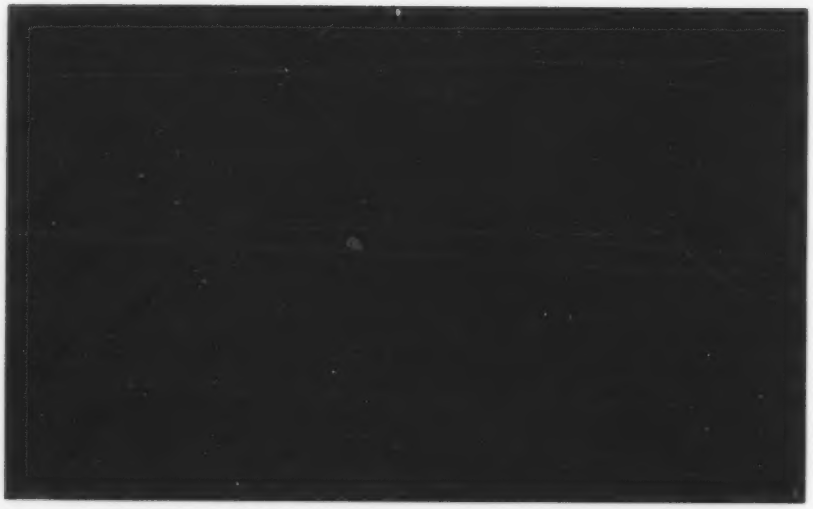

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Utilizing Shelve Slots: Sufficiency Conditions for Some Easy Instances of Hard Problems

by

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RÉSUMÉ

Dans ce rapport, nous présentons un ensemble minimal de conditions suffisantes pour assurer l'existence d'une solution à un système d'équations linéaires diophantiennes non négatives. Plus spécifiquement, étant donné un ensemble fini d'éléments $U = \{u_1, u_2, ..., u_n\}$, des "tailles" $v_i \equiv v(u_i) \in Z^+$, telles que $v_i \neq v_j$ si $i \neq j$, des "fréquences" $a_i \equiv a(u_i) \in Z^+$ et un nombre entier positif (longueur) $L \in Z^+$ vérifiant les conditions:

(i) $L = \prod_{j=1}^{n} p_j$ $(p_j \in Z^+ \forall_j, p_j \neq p_l \text{ si } j \neq l)$ et $v_i = \prod_{j \in A_i} p_j$, $A_i \subseteq \{1,2,...,n\}$ pour $i = 1,...,n$;

(ii) $\left(A_i \setminus \left\{\cap_{j=1}^{n} A_j\right\}\right) \cap \left(A_l \setminus \left\{\cap_{j=1}^{n} A_j\right\}\right) = \emptyset \ \forall i \neq l.$

Notons que $v_i \mid L$ (v_i divise L) pour chaque i.

Si, pour un $m \in Z^+$ donné, $\sum_{i=1}^{n} a_i v_i = mL$, alors nous démontrons que les conditions (i) et (ii) sont des conditions suffisantes pour l'existence d'un ensemble de nombres entiers $\{b_{11}, b_{12}, ..., b_{1m}, b_{21}, ..., b_{nm}\} \subseteq N$ tel que $\sum_{j=1}^{m} b_{ij} = a_i, i = 1,...n$ et $\sum_{i=1}^{n} b_{ij} v_i = L, j = 1,...,m$. Nous présentons un certain nombre de cas spéciaux de problèmes NP-complets qui sont alors décidés en temps polynomial.

Mots-clés: NP-complet, solvabilité polynomiale, BIN PACKING, SUBSET SUM, PARTITION.

ABSTRACT

In this paper we present a minimal set of conditions sufficient to assure the existence of a solution to a system of nonnegative linear Diophantine equations. More specifically, suppose we are given a finite item set $U = \{u_1, u_2, ..., u_n\}$ together with a "size" $v_i \equiv v(u_i) \in Z^+$, such that $v_i \neq v_j$ for $i \neq j$, a "frequency" $a_i \equiv a(u_i) \in Z^+$ and a positive integer (shelf length) $L \in Z^+$ with the following conditions:

(i) $L = \prod_{j=1}^{n} p_j$ $(p_j \in Z^+ \forall_j, p_j \neq p_l \text{ for } j \neq l)$ and $v_i = \prod_{j \in A_i} p_j$, $A_i \subseteq \{1,2,...,n\}$ for $i = 1,...,n$;

(ii) $\left(A_i \setminus \left\{\cap_{j=1}^{n} A_j\right\}\right) \cap \left(A_l \setminus \left\{\cap_{j=1}^{n} A_j\right\}\right) = \emptyset \ \forall i \neq l.$

Note that $v_i \mid L$ (divides L) for each i. If for a given $m \in Z^+$, $\sum_{i=1}^{n} a_i v_i = mL$ (i.e., the total size of all the items equals the total length of the shelf space), we prove that conditions (i) and (ii) are sufficient conditions for the existence of a set of integers $\{b_{11}, b_{12}, ..., b_{1m}, b_{21}, ..., b_{nm}\} \subseteq N$ such that $\sum_{j=1}^{m} b_{ij} = a_i, i = 1,...n$ and $\sum_{i=1}^{n} b_{ij} v_i = L, j = 1,...,m$ (i.e. m shelves of length L can be fully utilized). We indicate a number of special cases of well known NP-Complete problems which are subsequently decided in polynomial time.

Keywords: NP-Complete, polynomial solvability, BIN PACKING, SUBSET SUM, PARTITION.

1. INTRODUCTION

The advent of sophisticated computer technology (VLSI) in the telecommunication industry has led to the increased use of computerized subcomponents of network elements, e.g. switches and transmitters. These components are modular in design and may be unplugged and replaced in case of failure or repair. Each modular component takes up an integer (≥ 1) number of slots in the shelves of the equipment bay of the network element. In many cases the shelves are of equal length containing a fixed number of slots into which the modular components can be inserted in any order.

It has been often noticed in industrial practice that whenever the total length - the number of slots required by components of the network element - equals an integer multiple m of a shelf length and each component length is a divisor of the shelf length, then it is conjuctured that the m shelves can be fully utilized. The purpose of this paper is to establish a more precise conditions on the relationship between the components lengths and the shelf length which assures such an outcome. Divisibility of the shelf length by the components' lengths is not sufficient by itself.

More formally, let L denote the shelf length (number of slots), m be the number of shelves (integer ≥ 1), v_i denote the length ('size') of component $i, i = 1, 2, \cdots, k, v_i \neq v_j$ for $i \neq j$. Let a_i denote the number of units (the 'frequency') of component i required for the network element in question. Given (i) $L = \prod_{j=1}^{n} p_j (p_j \in Z^+ \ \forall j, p_j \neq p_l$ for $j \neq l)$ and $v_i = \prod_{j \in A_i} p_j, A_i \subseteq \{1, 2, \cdots, n\}$, and (ii) $(A_i \setminus \{\cap_{j=1}^{k} A_j\}) \cap (A_l \setminus \{\cap_{j=1}^{k} A_j\}) = \emptyset \ \forall i \neq l$. Given that the total number of slots available, mL, equals the total length of the components required ($mL = \sum_{i=1}^{k} a_i v_i = mL$), then we can insert all the components in the m shelves where each component is assigned to exactly one shelf if conditions (i) - (ii) hold.

Illustration: let $L = 30, m = 3, k = 4$ and the components have the following lengths and frequencies: $v_1 = 15, a_1 = 1; v_2 = 10, a_2 = 3; v_3 = 6, a_3 = 5; v_4 = 3, a_4 = 5$. An assignment of components to shelves is shown in Figure 1.

Figure 1 about here

If we drop the divisibility requirement of the shelf length L by the length of each individual component, then there might be no solution which fully utilizes the total shelf space.

Example 1 : Let $L = 20, m = 3, k = 4$, and the components have lengths: $v_1 = 15, a_1 = 1; v_2 = 10, a_2 = 3; v_3 = 6, a_3 = 2; v_4 = 3, a_4 = 1$. Clearly, $\sum_{i=1}^{4} v_i a_i = mL = 60$ but after inserting the first component ($v_1 = 15$), no remaining component combination will fill exactly the remaining shelf space of 5.

If we do not require that $L = \prod_{j=1}^{n} p_j$ and each $v_i = \prod_{j \in A_i} p_j$, then again we cannot be sure that the shelves can be fully utilized as illustrated by Example 2.

Example 2 : Let $L = 60, m = 2$ and $k = 4$ and let $v_1 = 30, a_1 = 1; v_2 = 20, a_2 = 2; v_3 = 12, a_3 = 4; v_4 = 2, a_4 = 1$. Clearly, $\sum_{i=1}^{4} v_i a_i = mL = 120$, each $v_i | L$, but the items cannot be fitted into the two shelves. $L = \prod_{j=1}^{3} p_j = 5 \times 4 \times 3$ but we cannot express 30 or 2 as a product of the p_j's.

Even the above example does not capture all the relations between L and the v_i's which consititute sufficient conditions for full utilization of m shelves of length L.

Example 3 : Let $L = 30, m = 2$ and $k = 4$, and let the components have lengths and frequencies $v_1 = 15, a_1 = 1; v_2 = 10, a_2 = 2; v_3 = 6, a_3 = 4; v_4 =$

$1, a_4 = 1$. It is easy to see that the components do not fit into the two shelves. $L = 5 \times 3 \times 2 \times 1$ and $15 = 5 \times 3 \times 1, 10 = 5 \times 2 \times 1, 6 = 3 \times 2 \times 1$, and $1 = 1$ but $(A_2 \setminus \{\cap_{i=1}^{4} A_i\}) = \{5, 2\} \cap \{3, 2\} = (A_3 \setminus \{\cap_{i=1}^{4} A_i\})$ equals $\{2\}$ and not $\emptyset$.

Example 4 : If we take an example similar to Example 3, i.e, $L = 30 = 5 \times 3 \times 2 \times 1, m = 2$ and $k = 3$, and let the components have lengths and frequencies $v_1 = 15, a_1 = 1; v_2 = 6, a_2 = 3; v_3 = 3, a_3 = 9$. In this example the conditions (i), and (ii) are satisfied ($v_1 = 15 = 5 \times 3 \times 1, v_2 = 6 = 3 \times 2 \times 1, v_3 = 3 = 3 \times 1$) and it is easy to see that the two shelves can be fully utilized.

In this paper we prove that conditions (i), and (ii), constitute a minimal set of sufficient conditions for the existence of a solution where each component is assigned to exactly one shelf and the entire shelve space is utilized (i.e., we prove that the answer is YES for the decision version of this problem).

We refer to this problem of space utilization as the Shelf-Packing Problem - (SPP) and to the requirement that each $v_i | L, i = 1, \cdots, n$, as the divisibility property.

Before proving that the SPP has a solution which fully utilizes the shelf space given conditions (i) - (ii) are satisfied, we present a number of equivalent problem statements and indicate the implications of this proof for other combinatorial optimization problems.

We start with a mathematical programming model version of the problem followed by its representation in terms of network flow with gains. Then we present the SPP as a special cases of decision problems such as PARTITION, BIN PACKING, and the SUBSET SUM problem.

Our literature search for similar work in combinatorial optimization or number theory has failed to come up with related work, except for Marcotte (1982,

1985), who proved that certain classes of knapsack inequalities, where the item sizes have the property of successive divisibility, give rise to cutting stock problems with an integer round-up property, i.e., easily solvable cutting stock problems. One other published work of similar nature is that of Coffman et al. (1987), where conditions are developed under which optimal bin packing solutions can be easily constructed.

2. PROBLEM STATEMENTS

2.1 Mathematical Programming Formulation of SPP

The SPP can be presented in terms of the existence of a feasible nonnegative integral solution to the following set of $n + m$ equations (a nonnegative linear diophantine system, see Schrijver, 1986):

$$\sum_{i=1}^{k} v_i X_{ij} = L \qquad j = 1, \cdots, m \qquad (1)$$

$$\sum_{j=1}^{m} X_{ij} = a_i \qquad i = 1, \cdots, k \qquad (2)$$

where X_{ij} is a nonnegative integral solution to (1) and (2). Clearly, $\sum_{j=1}^{m} \sum_{i=1}^{k} v_i X_{ij} = mL$.

The issue which we address in this paper is that of sufficiency conditions for the existance of a nonnegative integral solution to (1) and (2). We prove that (i) - (ii) are sufficient conditions and that they constitute a minimal set of such conditions.

2.2 Network Flow Formulation

Given a complete bipartite graph $K_{n,m} = G(N_1 \cup N_2, A)$ where N_1 are source nodes and N_2 are sink nodes, $|N_1| = k, |N_2| = m$ and A are the arcs from N_1 to

N_2. Each node $i \in N_1$ generates a flow of a_i and each sink node $j \in N_2$ absorbs a flow of L units. Each arc $e \in A$ originating from a node $i \in N_1$ has a flow multiplier of v_i associated with it. (See illustration in Figure 2.) The question of interest is: What are the sufficient conditions which assure the existence of integral flows given positive integral values for $v_i, a_i, i = 1, \cdots, k$, and L. We prove that (i) - (ii) are sufficient conditions for the existence of integral flows given that $\sum_{i=1}^{k} v_i a_i = mL$.

Figure 2 about here

2.3 Formulation of SPP as a PARTITION Problem

The PARTITION Problem can be stated as follows (Garey and Johnson, 1979):

INSTANCE: Finite set A and for each $a \in A$ a size $s(a) \in Z^+$ (the set of positive integers).

QUESTION: Is there a subset $A' \subset A$ such that

$$\sum_{a \in A'} s(a) = \sum_{a \in A \setminus A'} s(a)$$

Given $m = 2$ (i.e., $\sum_{a \in A} s(a) = 2L$) the PARTITION Problem has the exact form of SPP where $A = \{v_1^1, \cdots, v_{a_1}^1, v_1^2, \cdots, v_{a_2}^2, \cdots, v_1^n, \cdots, v_{a_n}^n\}$ and s is the identity function.

The PARTITION Problem is NP-Complete in the ordinary sense (see Garey and Johnson, 1979, for explansion of the relevant terminology). Proving that (i) - (ii) are sufficient conditions for the existence of such a subset $A' \subset A$ implies that our case of the PARTITION Problem is easy, i.e., belongs to P.

2.4 Formulating SPP as a BIN-PACKING Problem

The BIN-PACKING Problem is stated as follows:

INSTANCE: Finite set U of items, a size $s(u) \in Z^+$ for each $u \in U$, a positive integral bin capacity L, and a positive integer m.

QUESTION: Is there a partition of U into disjoint subsets $U_1, U_2, \cdots, U_m$ such that the sum of sizes of the items in each U_i is L or less ?

Clearly, if we take U equal to the set A and $s(u) \in Z^+$ as defined for the special case of the PARTITION Problem and require that the total size of items in U be equal to mL, then the BIN-PACKING Problem is reduced to the SPP.

The BIN-PACKING Problem is NP-Complete in the strong sense and NP-Complete in the ordinary sense for each fixed $m \geq 2$ (i.e., solvable in pseudo-polynomial time for fixed m). It is solvable in polynomial time for any fixed L by exhaustive search (Garey and Johnson, 1979).

By proving that conditions (i) - (ii) are sufficient in order to answer affirmatively the BIN-PACKING Problem question for this special case, we thus expand the set of easily solvable cases of BIN-PACKING.

2.5 The SUBSET SUM Problem

SUBSET SUM Problem can be stated as follows:

INSTANCE: Given a finite set A, a size $s(e) \in Z^+$ for each $e \in A$, and a positive integer L.

QUESTION: Is there a subset $A' \subseteq A$ such that

$$\sum_{e \in A'} s(e) = L$$

Again, if the sum $\sum_{e \in A} s(e) = mL$, for some integer $m \in Z^+$, and conditions (i) - (ii) are satisfied, then, the existance of an integral solution to the SPP

will imply an affirmative answer to the question posed for the SUBSET SUM Problem which is NP-Complete in the general case.

3. PROVING SUFFICIENCY CONDITIONS for the SPP SOLUTION

3.1 Special Case of the SPP where $\prod_i v_i = L$

In this section we define all the notation used subsequently in the proof of the sufficiency conditions for a special case of SPP.

Notation:

$N = \{0, 1, 2, \cdots\}$; $\quad Z^+ = \{1, 2, \cdots\}$

$\lfloor X \rfloor$ = the greatest integer $\leq X$; $\quad \lceil X \rceil$ = the least integer $\geq X$

$\cdot|\cdot$ means "divide" (e.g. $a|b$ means "a divides b").

$\exists$ means "there exists"; $\quad \forall$ means "for all"

One can easily prove that if $a, b, \in Z^+$ then

$$\lfloor a/b \rfloor \geq a/b - (b-1)/b = (a+1)/b - 1$$

For this special case of SPP where $\prod_i^n v_i = L$ we will denote $v_i \equiv p_i$ thus, in this entire subsection we write $L = p_1 p_2 \cdots p_n = \prod_i^n p_i$. Note that we do not assume the p_i's being coprime.

Lemma 1: If $n = 1$ or 2 then $\forall p_1, \cdots, p_n \in Z^+$ and $\forall a_1, \cdots, a_n \in N$ such that $p_1 < p_2 < \cdots < p_n$ and $\sum_{k=1}^n a_k p_k = mL$ where $m \geq 2$, $\quad \exists b_1, \cdots, b_n \in N$ such that $\sum_{k=1}^n b_k p_k = L$ and $b_k \leq a_k$ for $1 \leq k \leq n$.

Proof: (i) The case of $n = 1$ is trivial.

(ii) For $n = 2, a_1 p_1 + a_2 p_2 \geq 2L \Rightarrow a_1 p_1 \geq L$ or $a_2 p_2 \geq L \Rightarrow a_1 \geq p_2$

or $a_2 \geq p_1 \Rightarrow b_1 = p_2$ and $b_2 = 0$ or $b_1 = 0$ and $b_2 = p_1$ are solutions and the proof is complete.

□

Lemma 2: $\forall p_2, p_3 \in Z^+, \forall a_1, a_2, a_3 \in N$ such that $1 < p_2 < p_3$ and $a_1 + a_2p_2 + a_3p_3 \geq 2p_2p_3 \quad \exists b_1, b_2, b_3 \in N$ such that $b_1 + b_2p_2 + b_3p_3 = p_2p_3$ and $b_k \leq a_k$ for $k = 1, 2, 3$.

Proof : Case (1): If $a_1 + a_2p_2 \geq p_2p_3 + p_2 - 1$ then $\lfloor a_1/p_2 \rfloor + a_2 \geq (a_1 + 1)/p_2 - 1 + a_2 = (a_1 + a_2p_2 - p_2 + 1)/p_2 \geq p_3$.

Therefore $\exists c_1, c_2 \in N$ such that $0 \leq c_1 \leq \lfloor a_1/p_2 \rfloor, 0 \leq c_2 \leq a_2$ and $c_1 + c_2 = p_3$ and thus $c_1p_2 + c_2p_2 = p_2p_3$ with $0 \leq c_1p_2 \leq a_1$ and $0 \leq c_2 \leq a_2$. Therefore, $b_1 = c_1p_1, b_2 = c_2$ and $b_3 = 0$ is a solution.

Case 2: If $a_1 + a_2p_2 < p_2p_3 + p_2 - 1$ then $a_3p_3 \geq 2p_2p_3 - a_1 - a_2p_2 > p_2p_3 - p_2 + 1 > p_2p_3 - p_3 = (p_2 - 1)p_3$.

Therefore $a_3 \geq p_2$ and we can set $b_1 = b_2 = 0, b_3 = p_2$.

□

Lemma 3: $\forall n \in Z^+, n \geq 2; \forall a_1, \cdots, a_n \in N; \forall p_1, \cdots, p_n \in Z^+$, if

(a) $p_i \leq p_{i+1} \quad 1 \leq i \leq n-1$

(b) $\exists i, 1 \leq i \leq n$ such that $1 + a_i > p_{n-\lfloor i/n \rfloor}$ and

(c) $\sum_{k \neq i} \lfloor a_k/p_i \rfloor p_k + a_i \geq L/p_i$

then $\exists c_1, \cdots, c_n \in N$ such that

(1) $0 \leq c_k \leq \lfloor a_k/p_i \rfloor, k \neq i$

(2) $0 \leq c_i \leq a_i$

(3) $\sum_{k \neq i} c_kp_k + c_i = L/p_i$

Proof : The proof is constructive.

Case 1 : $a_i \geq L/p_i$

In this case a trivial solution is $c_k = 0 \quad \forall k \neq i$, and $c_i = L/p_i$

Case 2 : $a_i < L/p_i$

Construct the appropriate c_i's as follows:

Step 0 : set $c_k^0 = \lfloor a_k/p_i \rfloor \forall k \neq i$ and $c_i^0 = a_i$ then, by hypothesis (c)

$$\sum_{k \neq i} c_k^0 p_k + c_i^0 \geq L/p_i$$

In case $\sum_{k \neq i} c_k^0 p_k + c_i^0 = L/p_i$ then $\{c_k^0\}_{k=1}^n$ is a solution.

Suppose that $\sum_{k \neq i} c_k^0 p_k + c_i^0 > L/p_i$. In this case, we modify (successively) the initial $\{c_k^0\}_{k=1}^n$ values until we obtain the values $\{c_k\}_{k=1}^n$ for which the following holds: $\sum_{k \neq i}^n c_k p_k + a_i \leq L/p_i$.

Step 1 : (Procedure for generating the $\{c_k'\}_{k=1}^n$ values.)

Initially set $s = 0$

WHILE $\sum_{k \neq i}^n c_k^s p_k + c_i^s > L/p_i$

Find the largest index $r \neq i$ such that $c_r^s > 0$ and

set $s = s + 1, c_k^s = c_k^{s-1}; k \neq r, c_r^s = c_r^s - 1$

Continue

Let $s = s_0$ and $r = k_0$ be the final values of s and r. Note that existence of such $\{c_k\}_{k=1}^n$ values is assured since $c_k = 0$ for $k \neq i$ together with $c_i = a_i$ satisfy (1) and (2). Also by hypothesis of Case 2, since $a_i < L/p_i$, we get $\sum_{s \neq i}^n c_k p_k + a_i < L/p_i$.

The above procedure terminates with the following inequalities:

$$\sum_{k\neq i,k_0} c_k^{j_0} p_k + c_{k_0}^{j_0} p_{k_0} + c_i^{j_0} \leq L/p_i < \sum_{k\neq i,k_0} c_k^{j_0} p_k + (c_{k_0}^{j_0}+1) p_{k_0} + c_i^{j_0}$$

If the first inequality is satisfied with equality then set $\{c_k\}_{k=1}^n = \{c_k^{j_0}\}_{k=1}^n$ as the solution. If the inequality is strict, then let

$$\Delta \equiv \sum_{k\neq i,k_0} c_k^{j_0} p_k + (c_{k_0}^{j_0}+1) p_{k_0} + c_i^{j_0} - L/p_i > 0$$

Now $a_i > p_{k_0} - 1$ because if $i = n$, then $k_0 \neq n$ by definition of step s_0 and $a_i > p_{n-1} - 1 \geq p_{k_0} - 1$ by (a) and (b), and if $i \neq n$ then $a_i \geq p_n - 1 \geq p_{k_0} - 1$ again by (a) and (b).

Therefore, if we set $c_i = a_i - \Delta$ and $c_k = c_k^{j_0-1}, k \neq i$, then $\{c_k\}_{k=1}^n$ is a solution satisfying (1),(2), and (3), because $0 \leq c_k \leq \lfloor a_k/p_i \rfloor$ for $k \neq i$ since the $c_k^{j_0-1}$ already satisfy (1).

Also, $c_i = a_i - \Delta < a_i$ and

$$c_i = a_i - \Delta > p_{k_0} - 1 - p_{k_0} + L/p_i - [\sum_{k\neq i,k_0} c_k^{j_0} p_k + c_{k_0}^{j_0} p_{k_0} + c_i^{j_0}] \geq -1$$

which implies that $c_i > -1$, and hence $c_i \geq 0$. Thus c_i satisfies (2).

Finally,

$$\sum_{k\neq i,k_0} c_k p_k + c_i = \sum_{k\neq i,k_0} c_k^{j_0} p_k + (c_{k_0}^{j_0}+1) p_{k_0} + a_i - \Delta$$

$$= \sum_{k\neq i,k_0} c_k^{j_0} p_k + (c_{k_0}^{j_0}+1) p_{k_0} + a_i - [\sum_{k\neq i,k_0} c_k^{j_0} p_k + (c_{k_0}^{j_0}+1) p_{k_0} + a_i - L/p_i] = L/p_i$$

so that (3) is satisfied by $\{c_k\}_{k=1}^n$.

□

Theorem 1 : $\forall n, p_1, \cdots, p_n \in Z^+, \forall a_1, \cdots, a_n \in N$ such that $p_1 < p_2 < \cdots < p_n$ and $\sum_{k=1}^n a_k p_k = mL$ where $m \geq 2$, $\exists b_1, \cdots, b_n \in N$ such that $\sum_{k=1}^n b_k p_k = L$ and $b_k \leq a_k$ for $1 \leq k \leq n$.

Proof : We have already proven the theorem for the cases $n = 1, n = 2$, and $n = 3, p_1 = 1$. Next we prove it for all other n and p_is.

If $n = 3$ and $p_1 > 1$ then $L \geq 2p_2p_3 > (p_3 - 1)(p_1 + p_2)$;

If $n = 4$ and $p_1 > 1$ then $L \geq 6p_3p_4 > 3p_3p_4 > (p_4 - 1)(p_1 + p_2 + p_3)$;

If $n = 4, p_1 = 1$ and $p_2 > 2$ then $L \geq 3p_3p_4 > (p_4 - 1)(p_1 + p_2 + p_3)$;

If $n = 4, p_1 = 1$ and $p_2 = 2$ then $L = 2p_3p_4 \geq p_3p_4 + 3p_4 > (p_4 - 1)(3 + p_3) = (p_4 - 1)(p_1 + p_2 + p_3)$;

If $n \geq 5$ then $L \geq (n-2)!p_{n-1}p_n > (n-1)p_{n-1}p_n > (p_{n-1} - 1)\sum_{k=1}^{n-1} p_k$.

Thus, in all those cases we have $L > (p_n - 1)\sum_{k=1}^{n-1} p_k$. Consequently,

$$\sum_{k=1}^{n}(1 + a_k)p_k \geq \sum_{k=1}^{n} p_k + 2L >$$

$$> \sum_{k=1}^{n} p_k + (p_n - 1)\sum_{k=1}^{n-1} p_k + L > (p_1 + \cdots + p_{n-1})p_n + p_np_{n-1} = \sum_{k=1}^{n} p_kp_{n-\lfloor k/n \rfloor}$$

$(n - \lfloor k/n \rfloor$ is simply equal to $\max\{1, \cdots, n\} \backslash \{k\})$. Clearly then $\exists i, 1 \leq i \leq n$ such that $1 + a_i > p_{n-\lfloor i/n \rfloor}$.

Now for any $i, 1 \leq i \leq n$,

$$\sum_{k \neq i} \lfloor a_k/p_i \rfloor p_kp_i + a_ip_i \geq \sum_{k \neq i} (a_k + 1 - p_i)p_k + a_ip_i \geq 2L + (1 - p_i)\sum_{k \neq i} p_k \geq 2L + (1 - p_n)\sum_{k=1}^{n-1} p_k > L$$

because for $i \neq n$, we have $p_i < p_n$ and hence,

$$p_ip_n + (p_i - 1)(1 + \sum_{k \neq i,n} p_k) < p_ip_n + (p_n - 1)(1 + \sum_{k \neq i,n} p_k)$$

so that $(p_i - 1)\sum_{k \neq i} p_k < (p_n - 1)\sum_{k \neq n} p_k$.

$\forall n$, then, $\exists i, 1 \leq i \leq n$, such that $(1 + a_i) > p_{n-\lfloor i/n \rfloor}$ and $\sum_{k \neq i} \lfloor a_k/p_i \rfloor p_k + a_i > L/p_i$

From Lemma 3, $\exists c_1, c_2, \cdots, c_n \in N$ such that $0 \leq c_k \leq \lfloor a_k/p_i \rfloor$ for $k \neq i$, $0 \leq c_i \leq a_i$ and $\sum_{k \neq i} c_k p_k + c_i = L/p_i$

Thus, $\sum_{k \neq i}(c_k p_i)p_k + c_i p_i = L$ with $0 \leq c_k p_i \leq a_k$ for $k \neq i$ and $0 \leq c_i \leq a_i$. Therefore , set $b_k = c_k p_k, k \neq i$ and $b_i = c_i$.

□

Theorem 2 :$\forall n, p_1, \cdots, p_n \in Z^+$, and $\forall a_1, \cdots, a_n \in N$ such that $\sum_{k=1}^{n} a_k p_k = mL$, where $m \geq 2$, $\exists b_1, \cdots, b_n \in N$ such that $\sum_{k=1}^{n} b_k p_k = L$ and $b_k \leq a_k$ for $1 \; eq k \leq n$.

Proof : Assume $p_1 \leq \cdots \leq p_n$; and let $k_1, \cdots, k_r \in Z^+$ such that $0 < k_1 < \cdots < k_r = n$ and $p_1 = \cdots = p_{k_1} < p_{k_1+1} = \cdots = p_{k_2} < \cdots < p_{k_{r-1}+1} = \cdots = p_{k_r} = p_n$

Let $k_0 = 0$ and denote by $L' = p_{k_1} \cdots p_{k_r}$.

Now,

$$mL' \leq mL = \sum_{k=1}^{n} a_k p_k = \sum_{i=1}^{r} \sum_{j=k_{i-1}+1}^{k_i} a_j p_j = \sum_{i=1}^{r} p_{k_i} [\sum_{j=k_{i-1}+1}^{k_i} a_j]$$

Thus, from Theorem 1, $\exists c_{k_1}^{(1)}, \cdots, c_{k_r}^{(1)} \in N$ such that $\sum_{i=1}^{r} c_{k_i}^{(1)} p_{k_i} = L'$ and $c_{k_i}^{(1)} \leq \sum_{j=k_{i-1}+1}^{k_i} a_j$ for $1 \leq i \leq r$, so that $m[(L/L')-1]L' = \sum_{i=1}^{r} p_{k_i} [\sum_{j=k_{i-1}+1}^{k_i} a_j - c_{k_i}^{(1)}]$.

By repeated application of Theorem 1, $\exists c_{k_i}^{(l)}, i = 1, \cdots, r; s = 2, \cdots, L/L'$ such that

$$0 \leq c_{k_i}^{(s)} \leq [(\sum_{j=k_{i-1}+1}^{k_i} a_j) - (c_{k_i}^{(1)} + \cdots + c_{k_i}^{(s-1)})]$$

and $\sum_{i=1}^{r} c_{k_i}^{(s)} p_{k_i} = L'$

For $i = 1, \cdots, r$ let $c_i = c_{k_i}^{(1)} + \cdots + c_{k_i}^{L/L'}$

$0 \leq c_i \leq \sum_{j=k_{i-1}+1}^{k_i} a_j$ for $1 \leq i \leq n$ and $\sum_{i=1}^{r} c_i p_{k_i} = L/L' \times L' = L$

Now for each $i = 1, \cdots, r$, take $b_{k_i} = c_i$ and $b_k = 0$ for $k \neq k_i$. Thus, $0 \leq b_k \leq a_k, k = 1, \cdots, n$, and $\sum_{k=1}^{n} b_k p_k = L$

□

Corollary 1 : $\forall m, n, p_1, \cdots, p_n \in Z^+, \forall a_1, \cdots, a_n \in N$ such that $\sum_{k=1}^{n} a_k p_k = mL$, $\exists b_i^{(j)} \in N, i = 1, \cdots, n, j = 1, \cdots, m$, such that $\sum_{i=1}^{n} b_i^{(j)} p_i = L$, for $j = 1, \cdots, m$, and $\sum_{j=1}^{m} b_i^{(j)} = a_i$ for $i = 1, \cdots, n$.

Proof : Either $\sum_{k=1}^{n} a_k p_k = L$ (the trivial case) or $\sum_{k=1}^{n} a_k p_k = mL \geq 2L$. In the second case, the $b_i^{(j)}$'s are defined using Theorem 2 and induction on j.

□

Up to this point we have proven that the SPP has a solution in the special case where the product of the components' lengths equals the shelf length L. In the next section we generalize this result.

3.2 A More General Version of the SPP

In this section we extend the sufficiency conditions proving the existence of an SPP solution in the case where the lengths of each component $v_i, i = 1, \cdots, k$, are divisors of L but their product $\prod_{i=1}^{k} v_i \geq L$.

We assume in general that $\sum_{i=1}^{k} a_i v_i = mL, m \in \{2, 3, \cdots\}$ and $k \in \{2, 3, \cdots\}$.

Following the notation from the previous section we write $L = \prod_{i=1}^{n} p_i$ (where $p_1, \cdots, p_n \in Z^+$). Let $v_i = \prod_{j \in A_i} p_j$, for $i = 1, \cdots, k$, where $A_i \subseteq \{1, 2, \cdots, n\}$, and $(A_s \setminus \{\cap_{j=1}^{k} A_j\}) \cap (A_l \setminus \{\cap_{j=1}^{k} A_j\}) = \emptyset \ \forall s \neq l$ (i.e., the conditions (i) - (ii)).

Denote by $A^* = \cap_{i=1}^{k} A_i$

Theorem 3 : Suppose that $\sum_{i=1}^{k} v_i a_i = mL$ then $\exists b_i \in N$ such that

$0 \le b_i \le a_i, i = 1, \cdots, k$, and $\sum_{i=1}^{k} v_i b_i = L$.

Proof :

$$\sum_{i=1}^{k} v_i a_i = mL = \sum_{i=1}^{k} a_i \prod_{j \in A_i} p_j = \sum_{i=1}^{k} a_i \prod_{j \in A^*} p_j \prod_{j \in A_i \setminus A^*} p_j$$

Thus,

$$\sum_{i=1}^{k} a_i \prod_{j \in A_i \setminus A^*} p_j = mL / (\prod_{j \in A^*} p_j)$$

Note that in case $\cup_{i=1}^{k} A_i \neq \{1, \cdots, n\}$, we add a trivial element of $0 \times \prod_{j \notin \cup A_i} p_j$ to the left side of the expression above, and

$$L = (\prod_{j \in A^*} p_j) \times (\prod_{i=1}^{k} \prod_{j \in A_i \setminus A^*} p_j) \times (\prod_{j \notin \cup_{i=1}^{k} A_i} p_j)$$

By Corollary 1 there exist $b_i, 0 \le b_i \le a_i, i = 1, \cdots, k$, such that

$$\sum_{i=1}^{k} b_i \prod_{j \in A_i \setminus A^*} p_j = L / (\prod_{j \in A^*} p_j)$$

or

$$\sum_{i=1}^{k} b_i \prod_{j \in A_i} p_j = L = \sum_{k=1}^{k} b_i v_i$$

□

Note that condition (ii) is necessarily satisfied in the case $k = 2$.

4. Conclusion

We have addressed the problem of proving sufficiency conditions for the existence of a solution (a YES answer) to the Shelf Packing Problem (SPP). The motivation for this work originated from a real-world setting in telecommunication hardware. Consequently, we solved a number of special cases of well known NP-complete problems.

The main focus of this paper is on proving that conditions (i) - (ii) together with the condition that the totality of the elements' space equals the total shelf space, are sufficient conditions for the existence of a solution whereby each element is assigned a shelf space. It is clear that those conditions are minimal in the sense that for any relaxation one can produce a counterexample. The proof is constructive, involved, and broken into cases.

The relationship of our results to special cases of some very well known NP-complete problems gives a broader interpretation to the proof. In particular, we showed that special cases of diophantine systems, bipartite networks with gains, PARTITION, BIN-PACKING, and SUBSET SUM, can be reduced to the SPP. Thus, this work expands the present status of solvable classes of hard combinatorial problems in a novel fashion.

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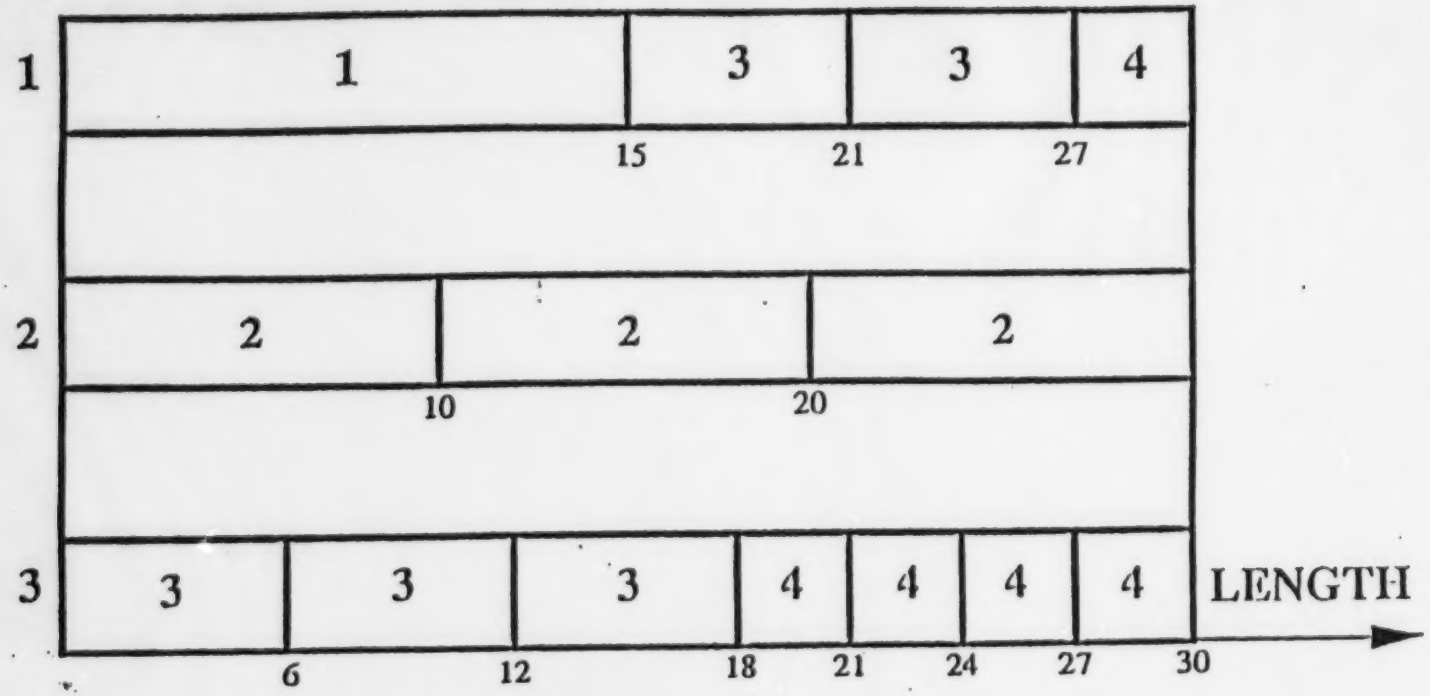

Figure 1: 3 shelf problem and its solution

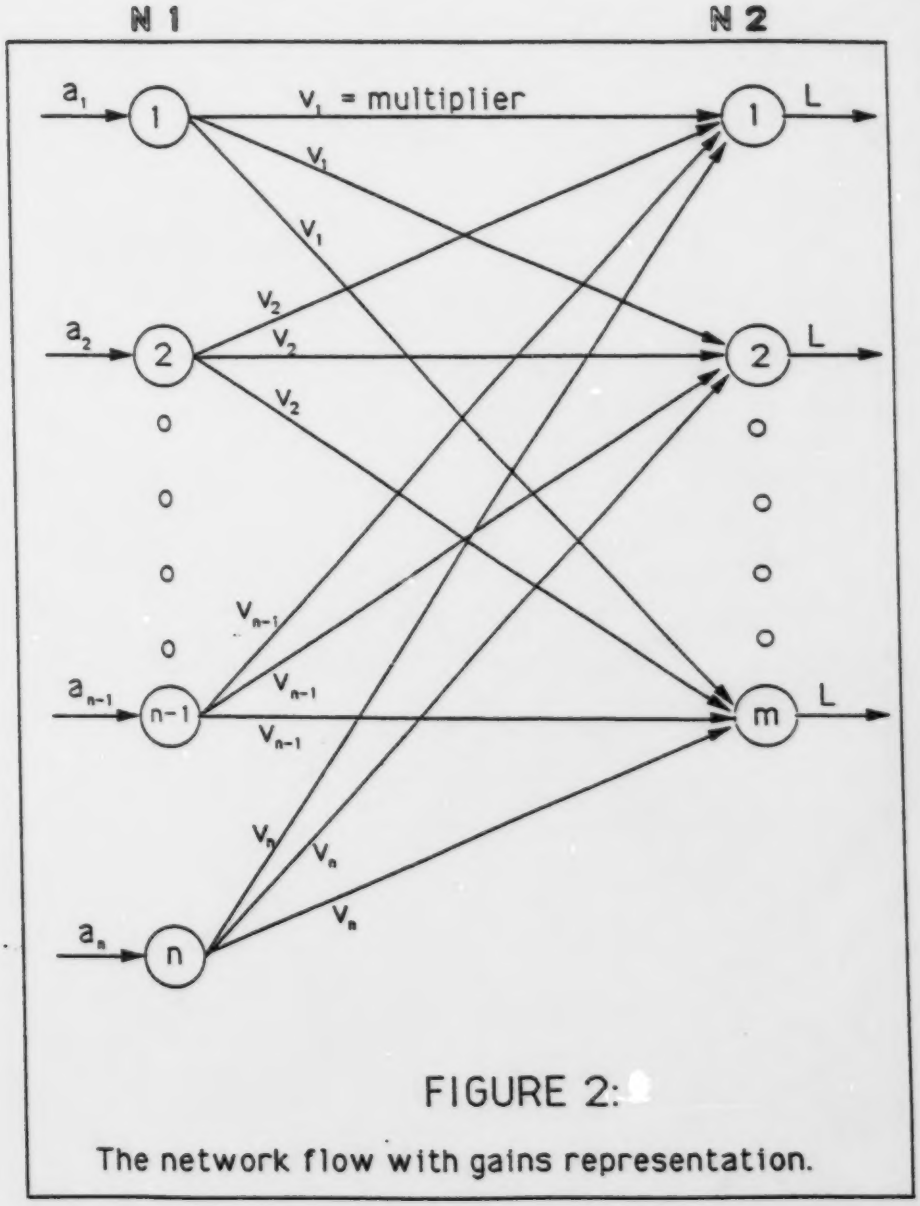

FIGURE 2:

The network flow with gains representation.